Contributions to Seymour's Second Neighborhood Conjecture

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Abstract

Let D be a simple digraph without loops or digons (i.e. if $(u, v) \in E(D)$, then $(v, u) \notin E(D)$). For any $v \in V(D)$ let $N_1(v)$ be the set of all vertices at out-distance 1 from v and let $N_2(v)$ be the set of all vertices at out-distance 2. We provide sufficient conditions under which there must exist some $v \in V(D)$ such that $|N_1(v)| \leq |N_2(v)|$, as well as examine properties of a minimal graph which does not have such a vertex. We show that if one such graph exists, then there exist infinitely many strongly-connected graphs having no such vertex.

1 Introduction

For the purposes of this article, we consider only simple nonempty digraphs (those containing no loops or multiple edges and having a nonempty vertex set), unless stated otherwise. We also require that our digraphs contain no digons, that is, if D is a digraph then $(u, v) \in E(D) \Rightarrow (v, u) \notin E(D)$. If i is a positive integer, we denote the i^{th} neighborhood of a vertex u in D by $N_{i,D}(u) = \{v \in V(D) | dist_D(u, v) = i\}$, where $dist_D(u, v)$ is the length of the shortest directed path from u to v in D (if there is no directed path from u to v, we set $dist_D(u, v) = \infty$). If D is clear from context, we simply write $N_i(u)$ and $dist(u, v)$. We also may wish to consider the i^{th} in-neighborhood of a vertex $N_{-i}(u) = \{v \in V(D) | dist(v, u) = i\}$. In addition, if $V' \subseteq V(D)$, we let $D[V']$ be the subgraph of D induced by V'.

Graph theorists will be familiar with the following conjecture due to Seymour (see [3]), now more than a decade old:

Conjecture 1.1 (Seymour's Second Neighborhood Conjecture)**.** *Let D be a directed graph. Then there exists a vertex $v_0 \in V(D)$ such that $|N_1(v_0)| \leq |N_2(v_0)|$.*

In 1995, Dean [3] conjectured this to be true when D is a tournament. Dean's Conjecture was subsequently proven by Fisher [5] in 1996. Further, in their 2001 paper Kaneko and Locke [6] showed Conjecture 1.1 to be true if the minimum outdegree of vertices in D is

less than 7, and Cohn, Wright, and Godbole [2] showed that it holds for random graphs almost always. And finally, in 2007 Fidler and Yuster [4] proved that Conjecture 1.1 holds for graphs with minimum out-degree $|V(D)|-2$, tournaments minus a star, and tournaments minus a sub-tournament. While over the years there have been several attempts at a proof of Conjecture 1.1, none of these have yet been successful.

For completeness, we introduce the related Caccetta-Häggkvist conjecture [1], which was posed in 1978:

Conjecture 1.2 (Caccetta-Häggkvist Conjecture)**.** *If D is a directed graph with minimum outdegree at least $|V(D)|/k$, then D has a directed cycle of length at most k.*

Conjecture 1.1 would imply the $k = 3$ case of Conjecture 1.2. Much work has been done on Conjecture 1.2, including an entire workshop in 2006 sponsored by AIM and the NSF, yet Conjectures 1.1 and 1.2 both remain open.

We do not seek to prove Conjecture 1.1 in this paper. Rather, we prove the conjecture for various classes of graphs. We then take a different tack and provide conditions that must be satisfied by any appropriately-defined minimal counterexample to Seymour's Second Neighborhood Conjecture. This provides tools with which the conjecture can be approached; in one direction it may aid in showing the nonexistence of such a graph, while in the other direction we restrict the search space of possible counterexamples.

2 Definitions

We begin our investigation by defining some useful terms.

Definition 2.1. Suppose that D is digraph and $u \in V(D)$. We say that u is *satisfactory* if $|N_1(u)| \leq |N_2(u)|$. Also, u is a *sink* if $|N_1(u)| = 0$. Note that a sink is trivially satisfactory.

Definition 2.2. Let $\mathcal{A} = \{D | D \textit{ is a simple directed graph with no satisfactory vertices}\}$ be the set of counterexamples to Seymour's Second Neighborhood Conjecture. Let $\mathcal{A}' = \{D \mid |E(D)| = \min_{H\in\mathcal{A}} |E(H)|\}$ be the set of graphs in $\mathcal{A}$ with the fewest number of edges. Finally, let $\mathcal{A}'' = \{D \mid |V(D)| = \min_{H\in\mathcal{A}'} |V(H)|\}$ be the set of graphs in $\mathcal{A}'$ with the fewest number of vertices. We will refer to any element of $\mathcal{A}''$ as a *minimal criminal*. Note that $\mathcal{A}''$ is empty if and only if Conjecture 1.1 is true.

Definition 2.3. Let D be a digraph. Suppose that $u \in V(D)$. We define $W_D(u) = \{v | dist(u, v) \neq \infty\}$ to be the *walkable neighborhood* of u with respect to D. If D is clear from context, we simply write $W(u)$.

Also define $A_{s,D}(u) = |N_1(u)| - |N_2(u)|$ to be the *anti-satisfaction* of u. As usual, if G is clear from context, we simply write $A_s(v)$. Notice that u is satisfactory if and only if $A_s(u) \leq 0$.

Definition 2.4. Again let D be a directed graph. Recall that a transitive triangle T is a directed graph on three nodes a, b, c such that $(a, b), (a, c), (b, c) \in E(T)$. If $(u, v) \in E(D)$, we say that edge (u, v) is the *base of a transitive triangle* if u and v share a common first neighbor; that is, $|N_1(u) \cap N_1(v)| \geq 1$.

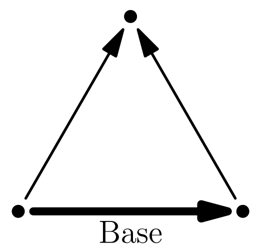

Figure 1 — Demonstration of an edge that is the base of a transitive triangle

If, for distinct $t, u, v, w \in V(D)$, we have that $(t,u),(u,w),(t,v),(v,w) \in E$ then we call $\{(t,u),(u,w),(t,v),(v,w)\}$ a *2-directed diamond*. We say the edges $(t,u),(t,v)$ are the *bases* of the 2-directed diamond.

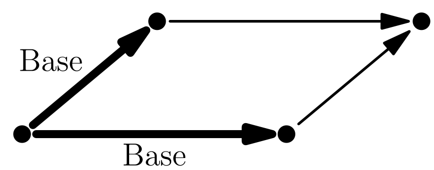

Figure 2 — Demonstration of the bases of a 2-directed diamond

We now have the tools to delve into our results.

3 Directed cycles and underlying girth

In this section we show that certain classes of graphs satisfy Seymour's Second Neighborhood Conjecture. The following theorem shows that directed cycles are necessary for a graph to be a counterexample to the conjecture.

Theorem 3.1. *If a digraph contains no directed cycles, then it must have a satisfactory vertex.*

Proof. Let D be a directed graph, and suppose that D contains no satisfactory vertices. Then D has no sink, as noted in Definition 2.1. It is a well-known fact that a graph with no sinks has a directed cycle. We include the standard proof, however, since the same technique will be useful to us later: Because D has sink, for $v \in V(D)$, $|N_1(v)| > 0$. Pick an arbitrary vertex $v_0 \in V(D)$, and consider the sequence $\{v_i\}_{i=0}^{|V(D)|}$ defined recursively by $v_{i+1} \in N_1(v_i)$ for $i \geq 0$. By the Pigeonhole principle, there exist some $r \neq s$ such that $v_r = v_s$. Then we note that the sequence of edges $(v_r, v_{r+1}), (v_{r+1}, v_{r+2}), \ldots, (v_{s-1}, v_s = v_r)$ defines a dicycle in D, thus completing our proof. $\square$

The following theorem provides another sufficient condition for a graph to contain a satisfactory vertex:

Theorem 3.2. *Let D be a directed graph containing no transitive triangles. Then D contains a satisfactory vertex.*

Proof. Let $v_0 \in V(D)$ have the minimal out-degree in D. If $|N_1(v_0)| = 0$, then v_0 is a sink and hence a satisfactory vertex. Otherwise, let $v_1 \in N_1(v_0)$. By construction, we have that $|N_1(v_1)| \geq |N_1(v_0)|$. Furthermore, D contains no transitive triangles, so $|N_1(v_0) \cap N_1(v_1)| = 0$. Thus, $|N_2(v_0)| \geq |N_1(v_1)| \geq |N_1(v_0)|$, and by definition v_0 is satisfactory. □

Remark. Recall that the girth of a undirected graph is the length of its shortest cycle. Theorem 3.2 shows that any counterexample to Conjecture 1.1 must have underlying girth of exactly 3.

4 Minimal Criminals

To this point, we have been showing that classes of graphs satisfy Conjecture 1.1. In this section we reverse course and explore necessary properties of the minimal criminal graphs of $\mathcal{A}''$ from Definition 2.2. If Seymour's Second Neighborhood Conjecture is true, then our goal should be to derive such strong constraints on the graphs of $\mathcal{A}''$ that a contradiction is obtained. On the other hand, if the conjecture is false, then our goal is to find necessary or sufficient conditions for a graph to be in $\mathcal{A}''$; we provide a number of necessary conditions here.

Theorem 4.1. *If $\mathcal{M} \in \mathcal{A}''$, we have the following:*

1. *$\mathcal{M}$ is strongly connected.*
2. *For each $u \in V(\mathcal{M})$, $A_s(u) \in \{1, 2\}$.*
3. *For every edge $e = (u, v) \in E(\mathcal{M})$, there exists a path of length 1 or 2 avoiding e from u to all but at most 1 element of $\{v\} \cup N_1(v)$.*
4. *Every edge of $\mathcal{M}$ is the base of either a transitive triangle or a 2-directed diamond.*
5. *Suppose that $e = (u, v) \in E(\mathcal{M})$ and $|N_1(u)| \leq |N_1(v)|$. Then e must be the base of at least $|N_1(v)| - |N_1(u)| + 1$ transitive triangles and the base of at least $|N_1(v)| - |N_1(u)| + 1$ 2-directed diamonds.*
6. *For any vertex $u \in V(\mathcal{M})$, there exists a vertex $v \in N_{-1}(u)$ such that $A_s(v) = 1$.*
7. *There exists a directed cycle in $\mathcal{M}$ such that every vertex on the cycle has anti-satisfaction of exactly 1.*

Proof. **Proof of 1:** Recall that a directed graph is strongly connected if there exists a directed path between any two of its vertices. Pick an arbitrary vertex u from the vertex set of $\mathcal{M}$. Now consider $\mathcal{M}' = \mathcal{M}[W(u)]$. We now pick an arbitrary vertex $v \in W(u)$. Clearly, $N_{1,\mathcal{M}}(v) \subseteq W(u)$ and $N_{2,\mathcal{M}}(v) \subseteq W(u)$. But this implies that $A_{s,\mathcal{M}'} = |N_{1,\mathcal{M}'}(v)| - |N_{2,\mathcal{M}'}(v)| = |N_{1,\mathcal{M}}(v)| - |N_{2,\mathcal{M}}(v)| = A_{s,\mathcal{M}}$, and hence v is satisfactory in $\mathcal{M}'$ if and only if v is satisfactory in $\mathcal{M}$. Since by definition $\mathcal{M}$ contains no satisfactory vertices, v cannot be satisfactory in $\mathcal{M}'$. Thus $\mathcal{M}'$ contains no satisfactory vertices. But $\mathcal{M}'$ is a subgraph of $\mathcal{M}$, and so by minimality of $\mathcal{M}$ we have that $\mathcal{M} = \mathcal{M}'$.

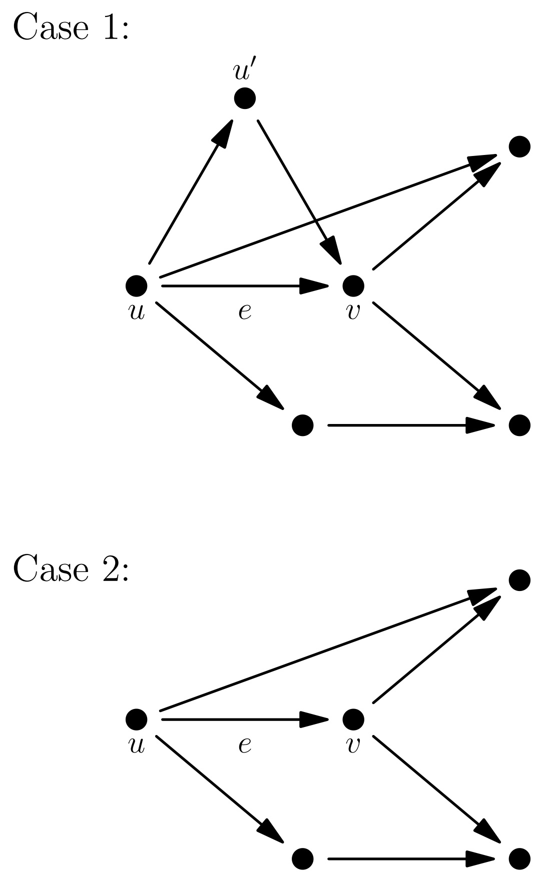

Figure 3 — Two possible cases resulting from deleting an edge from $\mathcal{M}$. In Case 1, there is a length 2 path from u to v, while in Case 2 no such path exists. Note that it is possible that deleting e will increase the size of u's second neighborhood, as shown in Case 1.

Proof of 2: Fix u and pick an arbitrary edge $e = (u, v) \in E(\mathcal{M})$. Consider the directed graph Z obtained by deleting e from $\mathcal{M}$. Since Z has fewer edges than $\mathcal{M}$, we have that Z contains a satisfactory vertex. For each vertex $w \in V(\mathcal{M})$, we note that $|N_{1,Z}(w)| = |N_{1,\mathcal{M}}(w)|$ unless $w = u$, in which case $|N_{1,Z}(u)| = |N_{1,\mathcal{M}}(u)| - 1$. Furthermore, we have that $|N_{2,Z}(w)| \leq |N_{2,\mathcal{M}}(w)|$, except if $w = u$, in which case we have that $|N_{2,Z}(u)| \leq |N_{2,\mathcal{M}}(u)| + 1$. (See Figure 3.)

Thus, we obtain that in Z for $w \neq u \in V(Z)$, $A_{s,Z}(w) \geq A_{s,\mathcal{M}}(w)$, and hence all vertices in Z besides u are not satisfactory. Thus by process of elimination we have that u is satisfactory in Z. Thus $0 \geq A_{s,Z}(u) = |N_{1,Z}(u)| - |N_{2,Z}(u)| \geq (|N_{1,\mathcal{M}}(u)| - 1) - (|N_{2,\mathcal{M}}(u)| + 1)$, and hence we have that $0 < A_{s,\mathcal{M}}(u) = |N_{1,\mathcal{M}}(u)| - |N_{2,\mathcal{M}}(u)| \leq 2$. Result 2 follows immediately.

Proof of 3: We see that $|N_{2,Z}(u)| \geq |N_{2,\mathcal{M}}(u)|$, since otherwise $A_{s,Z}(u) \leq 0$ and u is not satisfactory in Z, a contradiction. Consider now $X = N_{2,Z}(u) \setminus N_{2,\mathcal{M}}(u)$. We note that $X \subseteq \{v\}$, since v is the only vertex that could have been added to u's second neighborhood in Z (Case 1 in Figure 3). Thus we see that $|N_{2,\mathcal{M}}(u) \setminus N_{2,Z}(u)| \leq 1$, with equality only if $v \in N_{2,Z}(u)$.

Note that $N_{1,\mathcal{M}}(v) \subseteq N_{1,\mathcal{M}}(u) \cup N_{2,\mathcal{M}}(u)$. Let $Y = N_{1,\mathcal{M}}(u) \cap N_{1,\mathcal{M}}(v)$ and $Z = N_{2,\mathcal{M}}(u) \cap N_{1,\mathcal{M}}(v)$. For $y \in Y$, we clearly have a path of length 1 from u to y avoiding e (namely the

edge (u, y)). If $|N_{2,\mathcal{M}}(u) \setminus N_{2,Z}(u)| = 0$, then for $z \in Z$, we therefore have a path of length 2 from u to z in Z, and considering this path in $\mathcal{M}$ yields a path from u to z avoiding e. And finally, if $|N_{2,\mathcal{M}}(u) \setminus N_{2,Z}(u)| = 1$, then we have a path of length 2 from u to z in Z for all but 1 vertex in Z, and as before we have a corresponding path from u to z avoiding e. But in this case, there is a path of length 2 from u to v avoiding e, and hence we have obtained the desired result.

Proof of 4: Paths of length 1 from u to $v' \in N_1(v)$ yield transitive triangles with e as the base, and paths of length 2 from u to $v' \in \{v\} \cup N_1(v)$ yield 2-directed diamonds with e as one of the bases. By part 3, at least one of these structures exists, and hence we are done.

Proof of 5: Since $N_1(v) \setminus (N_1(u) \cap N_1(v)) \subseteq N_2(u)$, we have that $|N_2(u)| \geq |N_1(v)| - |N_1(u) \cap N_1(v)|$. But since $\mathcal{M}$ contains no satisfactory vertices, we have that $|N_2(u)| < |N_1(u)|$. By transitivity, we obtain $|N_1(v)| - |N_1(v) \cap N_1(u)| < |N_1(u)|$. It then follows that $|N_1(v)| - |N_1(u)| < |N_1(v) \cap N_1(u)|$, but $|N_1(v) \cap N_1(u)|$ is the number of transitive triangles having base e, so we have proved the first half of part 5.

To prove the second half of this part, we consider the following cases:

Case 1: Suppose there exists a vertex u' such that $(u, u'), (u', v) \in E(\mathcal{M})$. By part 3, we know that u must be connected to at least $|N_1(v)| - 1$ elements of $N_1(v)$ via a path of length 1 or 2 avoiding e. But we see that u is adjacent to at most $|N_1(u) - 2|$ vertices in $N_1(v)$. Subtracting, we see that u is connected via a path of length 2 avoiding e to at least $|N_1(v)| - 1 - (|N_1(u)| - 2) = (|N_1(v)| - |N_1(u)|) + 1$ vertices in $N_1(v)$; each of which yields a 2-directed diamond of which e is the base, which is the desired result.

Case 2: Suppose there is no such u'. Then again applying part 3, it must be that there exists a path of length 1 or 2 avoiding e to each vertex in $N_1(v)$. But u is adjacent to at most $|N_1(u)| - 1$ of these vertices, and as before we count that there is a path of length 2 avoiding e from u to at least $|N_1(v)| - (|N_1(u)| - 1) = |N_1(v)| - |N_1(u)| + 1$ vertices in $|N_1(v)|$. Since each of these paths yield a 2-directed diamond with e as the base, we are done.

Proof of 6: In $\mathcal{M}$, pick an arbitrary vertex u. Delete this vertex (and all edges incident with it) and label the resulting directed graph Z. Then in a similar manner to before, one of the vertices in $N_{-1,\mathcal{M}}(u)$ must be satisfactory in Z by vertex minimality of $\mathcal{M}$. Label this vertex t. Since $|N_{1,Z}(t)| = |N_{1,\mathcal{M}}(t)| - 1$, t is satisfactory, and $|N_{2,Z}(t)| \subseteq |N_{2,\mathcal{M}}(t)|$ (note that in contrast to deleting an edge, deleting a vertex does not allow any vertices to add vertices to their second neighborhoods), we see that we must have $|N_{2,Z}(t)| = |N_{2,\mathcal{M}}(t)|$. It is then necessary that $A_{s,\mathcal{M}}(t) = 1$. Since u was arbitrary, we have obtained the desired result.

Proof of 7: We apply the same technique as we used Theorem 3.1. We present a brief sketch of our proof: by part 5, each vertex in $\mathcal{M}$ has an in-neighbor having anti-satisfaction of exactly 1. If we begin at an arbitrary vertex and choose one of its in-neighbors having anti-satisfaction of exactly 1, do the same for the resulting vertex, and iterate this process, at some point we must arrive back at a vertex we have already visited, thus constructing a directed cycle of vertices having anti-satisfaction exactly 1. □

Finally, we show that there is not a finite nonzero number of strongly-connected counterexamples to the conjecture. That is, either the conjecture is true, or there are an infinite number of (non-isomorphic) strongly-connected graphs that violate Conjecture 1.1. This

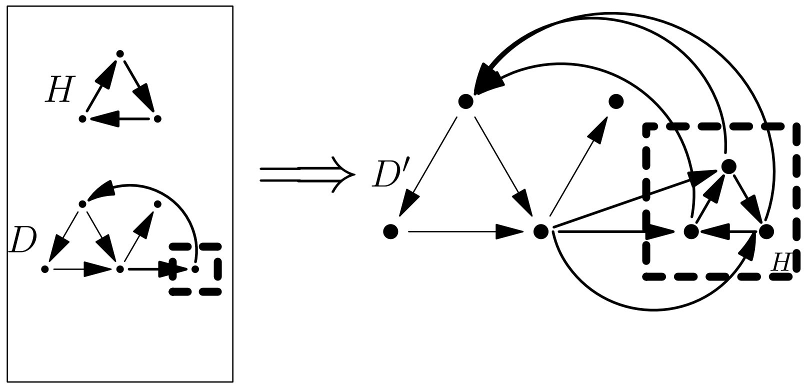

Figure 4 — A partial representation of the graph D', given D and H. We can think about D' as being made by replacing each vertex of D with a copy of H. Note that for clarity we replace only one vertex in the above picture.

is especially interesting in light of Part 1 of Theorem 4.1, which shows that all minimal criminals are strongly connected.

Theorem 4.2. *If Seymour's Second Neighborhood Conjecture is false, there are infinitely many non-isomorphic strongly-connected counterexamples to Seymour's Second Neighborhood Conjecture.*

Proof. Suppose that Seymour's Second Neighborhood Conjecture is false, and suppose that digraph D is any strongly-connected counterexample to Seymour's Second Neighborhood Conjecture. (By Part 1 of Theorem 4.1, such a D must exist.) Let H be any digraph satisfying the condition $\forall v \in V(H), A_s(v) \geq 0$; that is, all of H's vertices have nonnegative anti-satifaction. Note that any dicycle satisfies the relevant condition, and hence there exists a choice of H on any number n of vertices, $n \geq 3$.

We now construct a graph D' on $|V(D)| \cdot |V(H)|$ vertices such that D' is a counterexample to Seymour's Second Neighborhood Conjecture, thus proving our theorem. We define our graph D' as follows:

- $V(D') = V(D) \times V(H)$
- If $u = (d_1, h_1), v = (d_2, h_2) \in V(D')$, then $(u, v) \in E(D')$ if and only if either
    1. $d_1 = d_2$ and $(h_1, h_2) \in E(H)$, or
    2. $d_1 \neq d_2$ and $(d_1, d_2) \in E(D)$.

For any vertex $v = (d, h) \in V(D')$, we calculate that

$$|N_{1,D'}(v)| = |N_{1,H}(h)| + |V(H)| \cdot |N_{1,D'}(d)|,$$

by construction. Furthermore, we have that

$$|N_{2,D'}(v)| = |N_{2,H}(h)| + |V(H)| \cdot |N_{2,D'}(d)|.$$

We then calculate that

$$\begin{aligned} A_{s,D'}(v) &= |N_{1,D'}(v)| - |N_{2,D'}(v)| \\ &= (|N_{1,H}(h)| - |N_{2,H}(h)|) + |V(H)|(|N_{1,D'}(d)| - |N_{2,D'}(d)|). \end{aligned}$$

But by our choice of H, we have that $|N_{1,H}(h)| - |N_{2,H}(h)| \geq 0$, and by our choice of D we have that $|N_{1,D'}(d)| - |N_{2,D'}(d)| > 0$. Hence we obtain $A_{s,D'}(v) > 0$, thus implying that every vertex in D' has positive anti-satisfaction.

Furthermore, D' is strongly connected: fix $(d_1, h_1), (d_2, h_2) \in V(D')$. If $d_1 \neq d_2$, let $d_1, \delta_1, \ldots, \delta_i, d_2$ define a directed path in D from d_1 to d_2. Then

$$(d_1, h_1), (\delta_1, h_2), \ldots, (\delta_i, h_2), (d_2, h_2)$$

defines a directed path in D' from (d_1, h_1) to (d_2, h_2). If $d_1 = d_2$, let $d_3 \in N_{1,D}(d_1)$; we know that $(d_1, h_1), (d_3, h_2)$ are adjacent in D', and since $d_2 \neq d_3$ there is a path from (d_3, h_2) to (d_2, h_2) in D', the existence of a path from (d_1, h_1) to (d_2, h_2) follows.

By definition, we then have that D' is a strongly-connected counterexample to Seymour's Second Neighborhood Conjecture. □

5 Acknowledgements

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